Mark Scheme (Results)

Summer 2024

Pearson Edexcel GCE

In A Level Further Mathematics (9FM0)

Paper 02 Pure Mathematics

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Summer 2024
Question Paper Log P75683A
Publications Code 9FM0_02_2406_MS*

General Marking Guidance

- All candidates must receive the same treatment. Examiners must mark the first candidate in exactly the same way as they mark the last.
- Mark schemes should be applied positively. Candidates must be rewarded for what they have shown they can do rather than penalised for omissions.
- Examiners should mark according to the mark scheme not according to their perception of where the grade boundaries may lie.
- There is no ceiling on achievement. All marks on the mark scheme should be used appropriately.
- All the marks on the mark scheme are designed to be awarded. Examiners should always award full marks if deserved, i.e. if the answer matches the mark scheme. Examiners should also be prepared to award zero marks if the candidate's response is not worthy of credit according to the mark scheme.
- Where some judgement is required, mark schemes will provide the principles by which marks will be awarded and exemplification may be limited.
- When examiners are in doubt regarding the application of the mark scheme to a candidate's response, the team leader must be consulted.
- Crossed out work should be marked UNLESS the candidate has replaced it with an alternative response.

EDEXCEL GCE MATHEMATICS

General Instructions for Marking

1. The total number of marks for the paper is 75.

2. The Edexcel Mathematics mark schemes use the following types of marks:

- **M** marks: method marks are awarded for 'knowing a method and attempting to apply it', unless otherwise indicated.
- **A** marks: Accuracy marks can only be awarded if the relevant method (M) marks have been earned.
- **B** marks are unconditional accuracy marks (independent of M marks)
- Marks should not be subdivided.

3. Abbreviations

These are some of the traditional marking abbreviations that will appear in the mark schemes.

- bod – benefit of doubt
- ft – follow through
- the symbol √ will be used for correct ft
- cao – correct answer only
- cso - correct solution only. There must be no errors in this part of the question to obtain this mark
- isw – ignore subsequent working
- awrt – answers which round to
- SC: special case
- oe – or equivalent (and appropriate)
- dep – dependent
- indep – independent
- dp decimal places
- sf significant figures
- ∗ The answer is printed on the paper
- □ The second mark is dependent on gaining the first mark

4. For misreading which does not alter the character of a question or materially simplify it, deduct two from any A or B marks gained, in that part of the question affected.

5. Where a candidate has made multiple responses <u>and indicates which response they wish to submit</u>, examiners should mark this response.
If there are several attempts at a question <u>which have not been crossed out</u>, examiners should mark the final answer which is the answer that is the <u>most complete</u>.

6. Ignore wrong working or incorrect statements following a correct answer.

7. Mark schemes will firstly show the solution judged to be the most common response expected from candidates. Where appropriate, alternatives answers are provided in the notes. If examiners are not sure if an answer is acceptable, they will check the mark scheme to see if an alternative answer is given for the method used.

Question	Scheme	Marks	AOs
1(a)	$4\sinh^3 x + 3\sinh x \equiv 4\left(\frac{e^x - e^{-x}}{2}\right)^3 + 3\left(\frac{e^x - e^{-x}}{2}\right)$ $\equiv 4\left(\frac{e^{3x} - 3e^x + 3e^{-x} - e^{-3x}}{8}\right) + 3\left(\frac{e^x - e^{-x}}{2}\right)$	M1	2.1
	$\equiv \frac{e^{3x} - e^{-3x}}{2} \equiv \sinh 3x$ *	A1*	1.1b
		(2)	
(b)	$\sinh 3x = 19\sinh x \Rightarrow 4\sinh^3 x + 3\sinh x = 19\sinh x$ $\sinh 3x = 19\sinh x \Rightarrow 4\sinh^3 x - 16\sinh x = 0$ $4\sinh x\left(\sinh^2 x - 4\right) = 0$	M1	3.1a
	$\sinh x = 0 \Rightarrow x = 0$	B1	2.2a
	$\sinh^2 x = 4 \Rightarrow \sinh x = \pm 2$ $\Rightarrow x = \ln\left(\pm 2 + \sqrt{(\pm 2)^2 + 1}\right)$	M1	1.1b
	$x = \ln\left(2+\sqrt{5}\right)$ **or** $x = \ln\left(-2+\sqrt{5}\right)$ oe e.g. $x = -\ln\left(2+\sqrt{5}\right)$	A1	1.1b
	$x = \ln\left(2+\sqrt{5}\right)$ **and** $x = \ln\left(-2+\sqrt{5}\right)$ Alternatively, $x = \ln\left(\sqrt{5} \pm 2\right)$ oe e.g. $x = \pm\ln\left(2+\sqrt{5}\right)$ or $\frac{1}{2}\ln\left(9 \pm 4\sqrt{5}\right)$	A1	1.1b
		(5)	
		(7 marks)	

Notes

(a)

M1: Begins the proof by expressing sinh x correctly in terms of exponentials, substitutes and makes progress in cubing the bracket.

Award for obtaining an expression of the form $Ae^{3x} + Be^x + Ce^{-x} + De^{-3x}$ but terms do not need to be collected.

Note that $(e^{2x} - 2 + e^{-2x})(e^x - e^{-x}) = e^{3x} - e^x - 2e^x + 2e^{-x} + e^{-x} - e^{-3x}$

A1*: Fully correct proof with no errors. Must see $= \sinh 3x$ or e.g. LHS = RHS

(b)

M1: Uses part (a), collects terms and attempts to factorise or cancel sinh x.

This can be implied if they go straight from their cubic to writing **all** correct answers for sinh x including zero from their calculator.

B1: Deduces the root $x = 0$, allow ln1 but not $\ln\left[0 + \ln\sqrt{0+1}\right]$

M1: Proceeds to sinh $x = \alpha$ and uses the correct logarithmic form of arsinh to obtain at least one exact value for x

Alternatively, candidates proceed from sinh $x = \alpha$ to substitute the exponential form and then solve a 3TQ in e^x to obtain at least one exact value for x

A1: One correct non-zero solution

A1: Both correct non-zero solutions and no incorrect other solutions, but isw if they go on to evaluate these answers incorrectly.

Allow $x = \ln\left(\sqrt{5} \pm 2\right)$ for listing both solutions

Alternative if candidates substitute the exponential form at the start:
M1: Candidates substitutes the exponential form for each term and proceeds to find a four term cubic equation in $e^{2x} = 0$, which may not be correct.
B1: Deduces the root $x = 0$, allow ln1 but not $\ln\left[0+\ln\sqrt{0+1}\right]$
M1: They factorise their cubic, proceed to obtain exact values for e^{2x}, then take logs to obtain at least one exact value for x
If they go directly to decimal answers this will usually score M0 unless they recover to exact form.
A1: One correct non-zero solution
A1: Both correct non-zero solutions and no incorrect other solutions, but isw if they go on to evaluate these answers incorrectly.

This is how this would be awarded:

$$\frac{(e^{3x}-e^{-3x})}{2} = 19\left(\frac{e^{x}-e^{-x}}{2}\right)$$

$$e^{3x}-e^{-3x}-19e^{x}+19e^{-x}=0$$

$$e^{6x}-1-19e^{4x}+19e^{2x}=0$$ M1 (for cubic in $e^{2x} = 0$)

$$e^{6x}+19e^{4x}+19e^{2x}-1=0$$

$$\Rightarrow (e^{2x}-1)(e^{4x}-18e^{2x}+1)=0$$

$$e^{2x}=1,\ 9\pm4\sqrt{5}$$

$$2x=\ln 1,\ 2x=\ln\left(9\pm4\sqrt{5}\right)$$

$$x=0,\ x=\frac{1}{2}\ln\left(9\pm4\sqrt{5}\right)$$ B1, M1, A1, A1

Question	Scheme	Marks	AOs
2(a)	$\dfrac{\mathrm{d}\left(\dfrac{3-x}{6+x}\right)}{\mathrm{d}x}=\dfrac{-(6+x)-(3-x)}{(6+x)^2}$	M1 A1	3.1a 1.1b
	$\mathrm{f}(x)=\tanh^{-1}\left(\dfrac{3-x}{6+x}\right)\Rightarrow \mathrm{f}'(x)=\dfrac{1}{1-\left(\dfrac{3-x}{6+x}\right)^2}\times\dfrac{-9}{(6+x)^2}$	dM1	3.1a
	$=\dfrac{(6+x)^2}{36+12x+x^2-9+6x-x^2}\times\dfrac{-9}{(6+x)^2}=\dfrac{-9}{18x+27}=\dfrac{-1}{2x+3}$ *	A1*	2.1
		(4)	
	Alternative 1 for part (a)		
	$\dfrac{\mathrm{d}\left(\dfrac{3-x}{6+x}\right)}{\mathrm{d}x}=\dfrac{-(6+x)-(3-x)}{(6+x)^2}$	M1 A1	3.1a 1.1b
	$y=\tanh^{-1}\left(\dfrac{3-x}{6+x}\right)\Rightarrow \tanh y=\dfrac{3-x}{6+x}\Rightarrow \mathrm{sech}^2\, y\dfrac{\mathrm{d}y}{\mathrm{d}x}=\dfrac{-9}{(6+x)^2}$ $\dfrac{\mathrm{d}y}{\mathrm{d}x}=\dfrac{1}{1-\tanh^2 y}\times\dfrac{-9}{(6+x)^2}=\dfrac{1}{1-\left(\dfrac{3-x}{6+x}\right)^2}\times\dfrac{-9}{(6+x)^2}$	dM1	3.1a
	$=\dfrac{(6+x)^2}{36+12x+x^2-9+6x-x^2}\times\dfrac{-9}{(6+x)^2}=\dfrac{-9}{18x+27}=\dfrac{-1}{2x+3}$ *	A1*	2.1
		(4)	
	Alternative 2 for part (a)		
	$\mathrm{f}(x)=\tanh^{-1}\left(\dfrac{3-x}{6+x}\right)=\dfrac{1}{2}\ln\left(\dfrac{1+\dfrac{3-x}{6+x}}{1-\dfrac{3-x}{6+x}}\right)=\dfrac{1}{2}\ln\left(\dfrac{9}{3+2x}\right)$	M1 A1	3.1a 1.1b
	$\mathrm{f}(x)=\tanh^{-1}\left(\dfrac{3-x}{6+x}\right)\Rightarrow \mathrm{f}'(x)=\dfrac{1}{2}\times\dfrac{3+2x}{9}\times\dfrac{-18}{(3+2x)^2}$	dM1	3.1a
	$=\dfrac{-1}{2x+3}$ *	A1*	1.1b
		(4)	
	Alternative 3 for part (a)		
	$\tanh y=\mathrm{f}(x)=\left(\dfrac{3-x}{6+x}\right)\Rightarrow\dfrac{\mathrm{e}^{y}-\mathrm{e}^{-y}}{\mathrm{e}^{y}+\mathrm{e}^{-y}}=\dfrac{3-x}{6+x}$ or $\dfrac{\mathrm{e}^{2y}-1}{\mathrm{e}^{2y}+1}=\dfrac{3-x}{6+x}$ $\mathrm{e}^{2y}=\dfrac{9}{2x+3}$	M1 A1	3.1a 1.1b
	$y=\dfrac{1}{2}\ln\left(\dfrac{9}{2x+3}\right)\Rightarrow \mathrm{f}'(x)=\dfrac{1}{2}\times\dfrac{2x+3}{9}\times\dfrac{-18}{(2x+3)^2}$	dM1	3.1a
	$=\dfrac{-1}{2x+3}$ *	A1	1.1b
		(4)	

Notes
(a) **Do not allow misreads of tan*x* for tanh*x* in this question; this would be a maximum of M1A1 for part (a)**

M1: Attempts to use the quotient (or product) rule on $\frac{3-x}{6+x}$ to obtain an expression of the form

$$\frac{A(6+x)-B(3-x)}{(6+x)^2},\ B>0 \quad \textbf{or} \quad C(3-x)(6+x)^{-2}+D(6+x)^{-1}$$

Alternatively, candidates may also write $\frac{3-x}{6+x}$ as $-1+\frac{9}{6+x}$ and then differentiate to find an expression of the form $\frac{E}{(6+x)^2}$ which may be seen embedded in their working.

They may also write $\frac{3-x}{6+x}$ as $\frac{3}{6+x}-\frac{x}{6+x}$ and then differentiate to find an expression of the form $\frac{E}{(6+x)^2}$ oe

A1: Correct expression in any form.

dM1: A complete method to find the derivative using the chain rule to obtain

$$\frac{1}{1-\left(\frac{3-x}{6+x}\right)^2}\times\left(\text{their } \frac{-9}{(6+x)^2}\right)$$

This is dependent on the first M mark.

A1*: Reaches the printed answer with sufficient working and no errors, with at least one intermediate line.

Alternative 1:
M1: see main scheme
A1: see main scheme

dM1: A complete method to find the derivative using the chain rule:
Rearranges the equation and uses implicit differentiation. Must proceed from $\text{sech}^2\, y$ or equivalent to $1-\tanh^2 y$ and then substitute for $\tanh y$
This is dependent on the first M mark.
A1*: see main scheme

Alternative 2:

M1: Uses the logarithmic form of artanh to obtain $k\ln\left(\frac{1+\frac{3-x}{6+x}}{1-\frac{3-x}{6+x}}\right)$

A1: Correct simplified expression

dM1: A complete method to find the derivative to obtain

$$\text{f}'(x)=k\times\left(1\div\left(\text{their}\frac{9}{3+2x}\right)\right)\times\left(\text{their } \frac{9}{3+2x}\right)^2$$

Alternatively uses log rules to partition their expression and then differentiates each term.
This is dependent on the first M mark.

A1*: Reaches the printed answer with sufficient working and no errors.
Alternative 3:
M1: Takes tanh of both sides to obtain tanh$y$ in terms of x and expresses tanh$y$ correctly in terms of exponentials and makes e^{2y} the subject.
A1: A correct expression for e^{2y}

dM1: Rearranges their function to the form $k\ln\left(\frac{a}{bx+c}\right)$ **and** uses the chain rule to find a derivative of the form $f'(x)=\frac{1}{2}\times\frac{bx+c}{a}\times\frac{k}{(bx+c)^2}$ or uses log rules to partition their expression and then differentiates each term.
This is dependent on the first M mark.

A1*: Reaches the printed answer with sufficient working and no errors

(b)	$\left[f''(x)=\right]\ \frac{2}{(2x+3)^2}$	B1	1.1b
		(1)	
(c)	$f(0)=\tanh^{-1}\left(\frac{1}{2}\right)\left(=\frac{1}{2}\ln 3\right),\ f'(0)=-\frac{1}{3},\ f''(0)=\frac{2}{9}$	M1	1.1b
	$\left[f(x)\right]=f(0)+xf'(0)+\frac{x^2}{2}f''(0)$	M1	1.1b
	$=\ln\sqrt{3}-\frac{1}{3}x+\frac{1}{9}x^2$	A1	1.1b
		(3)	
			(8 marks)

Notes

(b)
B1: Correct second derivative in any form such as e.g. $\frac{2}{(4x^2+12x+9)}$ or $2(2x+3)^{-2}$

(c)
M1: Attempts at least two of the values of f(0), f′(0) and f″(0)
M1: Correct application of the Maclaurin series for **all** of $f(0)$, $f'(0)$ and $f''(0)$ where all are non-zero values. Substitutes their values into a correct expression. This mark is not dependent.

A1: Correct expansion.
Award if written as a single expression, or p, q and r written separately.
Ignore any extra terms written as powers of x^3 and above, isw when a correct answer is seen.

Question	Scheme	Marks	AOs
3(a)	Because the upper limit is infinite	B1	2.4
		(1)	
(b)	$\int \frac{1}{9x^2+16}\,dx = \frac{1}{12}\arctan\left(\frac{3x}{4}\right)$	M1 A1	3.1a 1.1b
	$\int_{\frac{4}{3}}^{\infty} \frac{1}{9x^2+16}\,dx = \frac{1}{12}\lim_{t\to\infty}\left[\arctan\left(\frac{3x}{4}\right)\right]_{\frac{4}{3}}^{t}$ $= \frac{1}{12}\left(\lim_{t\to\infty}\arctan\left(\frac{3t}{4}\right) - \arctan(1)\right)$	dM1	1.1b
	$= \frac{1}{12}\left(\frac{\pi}{2} - \frac{\pi}{4}\right) = \frac{\pi}{48}$	A1	2.1
		(4)	
		(5 marks)	

Notes

(a)

B1: Suitable explanation stating that one of the bounds/limits is infinite oe isw
e.g. "one of the limits is unbounded" or "the integral is unbounded"

Do not allow this mark if they only say the limit or integral is undefined, unless they go on to say it is undefined at infinity.
Commenting that the **function** is undefined at infinity is not enough to award this mark on its own, this question concerns the **limits.**
If the candidates state any extra incorrect comments about the **limits** withhold this mark e.g. not defined at $\frac{4}{3}$

(b)

M1: Integrates to obtain $\alpha \arctan(\beta x)$ where $\beta \neq 1$

A1: Correct integration, unsimplified or simplified.

dM1: Applies correct limits, "t" and $\frac{4}{3}$ with evidence of applying the infinite limit to obtain a non-zero value.

Allow with ∞ used as the limit (which may be implied by $\frac{\pi}{2}$)

A1: Correct value obtained with evidence of use of limiting process on the upper bound.
Withhold this mark if there is no evidence of using the limiting process. We must see as a minimum $\lim_{t\to\infty}$ oe at some stage in their work.

e.g. $\left[\frac{1}{12}\arctan\left(\frac{3x}{4}\right)\right]_{\frac{4}{3}}^{\infty} = \frac{1}{12}\left(\frac{\pi}{2} - \frac{\pi}{4}\right) = \frac{\pi}{48}$ would score dM1A0

Question	Scheme	Marks	AOs
4	$\frac{2}{(r+4)(r+6)} \equiv \frac{A}{r+4}+\frac{B}{r+6} \Rightarrow A=..., B=...$	M1	3.1a
	$\frac{2}{(r+4)(r+6)} \equiv \frac{1}{r+4}-\frac{1}{r+6}$	A1	1.1b
	$\sum_{r=1}^{n}\frac{2}{(r+4)(r+6)} = \sum_{r=1}^{n}\frac{1}{r+4}-\frac{1}{r+6}$ $\frac{1}{5}-\frac{1}{7}+\frac{1}{6}-\frac{1}{8}+\frac{1}{7}-\frac{1}{9}+....$ $+\frac{1}{n+2}-\frac{1}{n+4}+\frac{1}{n+3}-\frac{1}{n+5}+\frac{1}{n+4}-\frac{1}{n+6}$	M1	2.1
	$=\frac{1}{5}+\frac{1}{6}-\frac{1}{n+5}-\frac{1}{n+6}$	A1	2.2a
	$=\frac{1}{5}+\frac{1}{6}-\frac{1}{n+5}-\frac{1}{n+6}=\frac{11(n+5)(n+6)-30(n+6)-30(n+5)}{30(n+5)(n+6)}$	M1	1.1b
	$=\frac{n(11n+61)}{30(n+5)(n+6)}$	A1	1.1b
		(6)	
		(6 marks)	

Notes

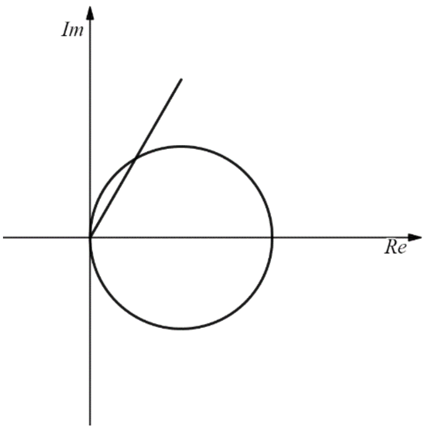
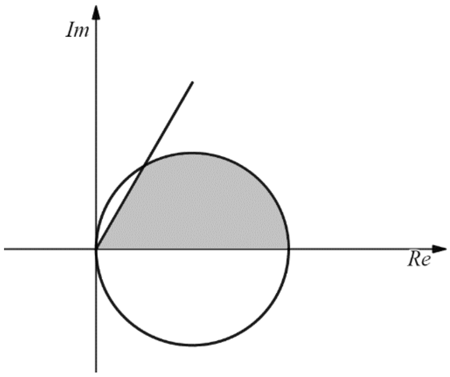
M1: Recognises the need to find partial fractions and applies a correct method leading to finding values for *A* and *B*
Allow a slip when finding the constants
A1: Correct partial fractions **seen** at any stage. Not just values for *A* and *B* listed

Note: Proof by induction will not score the next 4 marks.
M1: Starts the process of finding terms at the start and at the end, in order to establish the non-cancelling terms
Must have attempted a minimum of $r=1$, $r=2$, ... $r=n-1$ and $r=n$,this may be implied by their correct non-cancelling terms.
Follow through on their values of *A* and *B*. Look for

$r=1$ ® $\frac{A}{5}-\frac{B}{7}$ $\qquad$ $r=2$ ® $\frac{A}{6}-\frac{B}{8}$

$r=n-1$ ® $\frac{A}{n+3}-\frac{B}{n+5}$ $\qquad$ $r=n$ ® $\frac{A}{n+4}-\frac{B}{n+6}$

A1: Correct non-cancelling terms which may be listed separately.
Correct fractions from the beginning and end that do not cancel stated.
M1: Combines 'their' fractions of the form $p+\frac{q}{n+5}+\frac{r}{n+6}$ over a correct common denominator which does not need to be the lowest common denominator and obtains a quadratic expression in the numerator.
A1: Correct answer.
Note: If they start with $r=0$ the maximum they can score is M1A1M0A0M1A0

Question	Scheme	Marks	AOs
5(a)		M1 A1 M1 A1	1.1b 1.1b 1.1b 1.1b
		(4)	
(b)		B1ft	2.3
		(1)	

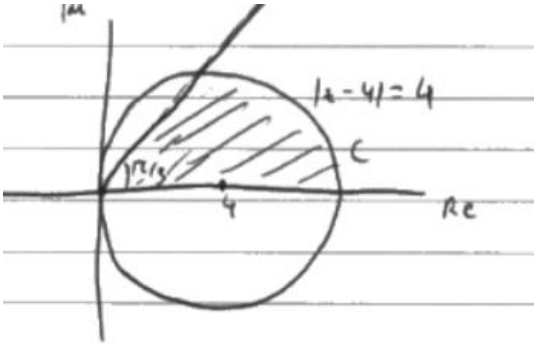
Notes

(a)
M1: Circle drawn with centre on the real axis
Look for real axis acting as a line of symmetry of the circle.
A1: Circle in the correct position with the imaginary axis as a tangent
Centre need not be labelled for either mark.
M1: Half line starting at the origin, must be in the first quadrant. Do not award if their line continues into the third quadrant
A1: Fully correct diagram that requires

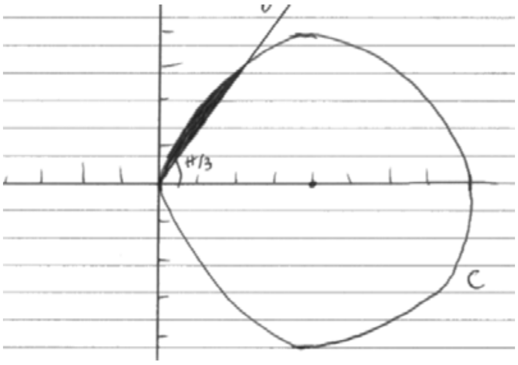
- A circle in the correct position
- A half line intersecting the circle at the origin and in the first quadrant
- The x coordinate of the intersection in the first quadrant must be to the left of the centre of the circle

(b)
B1ft: Shades the region in their circle above the real axis and below the half line
For this mark to be awarded their line must intersect the circle.

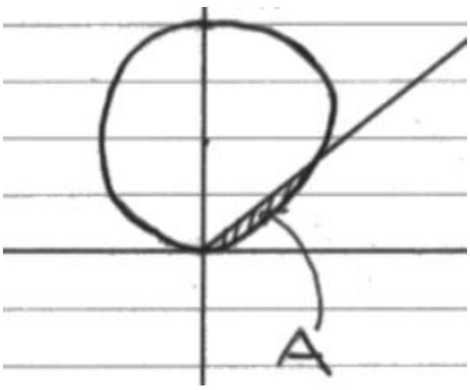
Q5 (a) and (b) examples



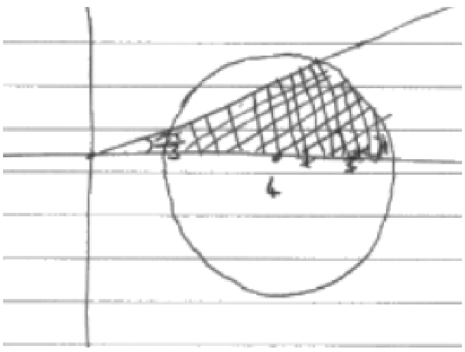
a) M1A1M1A1 b) B1



a) M1A1M1A1 b) B0



a) M0A0M1A0 b) B1



a) M1A0M1A0 b) B1

(c)	$(x-4)^2+y^2=16,\quad y=\sqrt{3}x \Rightarrow x^2-8x+16+3x^2=16 \Rightarrow x=...$ $x=2,\quad y=2\sqrt{3}$	M1 A1	3.1a 1.1b
	$\frac{1}{2}\pi\times 4^2-\left(\frac{1}{2}\times 4^2\times\frac{\pi}{3}-\frac{1}{2}\times 4^2\times\sin\frac{\pi}{3}\right)$ or $\frac{1}{2}\times 4^2\times\frac{2\pi}{3}+\frac{1}{2}\times 4\times 2\sqrt{3}$ or $\frac{1}{2}\times 4^2\times\frac{2\pi}{3}+\frac{1}{2}\times 4\times 4\times\frac{\sqrt{3}}{2}$	dM1	3.1a
	$=\frac{16}{3}\pi+4\sqrt{3}$	A1	1.1b
		(4)	
	Alternative 1 for part (c)		
	$x=4\cos\frac{\pi}{3},\quad y=4\sin\frac{\pi}{3}$	M1 A1	3.1a 1.1b
	$\frac{1}{2}\pi\times 4^2-\left(\frac{1}{2}\times 4^2\times\frac{\pi}{3}-\frac{1}{2}\times 4^2\times\sin\frac{\pi}{3}\right)$ or $\frac{1}{2}\times 4^2\times\frac{2\pi}{3}+\frac{1}{2}\times 4\times 2\sqrt{3}$ or $\frac{1}{2}\times 4^2\times\frac{2\pi}{3}+\frac{1}{2}\times 4\times 4\times\frac{\sqrt{3}}{2}$	dM1	3.1a
	$=\frac{16}{3}\pi+4\sqrt{3}$	A1	1.1b
		(4)	
	Alternative 2 for part (c)		
	Deduces $\frac{\pi}{3}$ means there is an equilateral triangle of length 4 4 and $\sin\frac{\pi}{3}$ oe are seen or used in their workings	M1 A1	3.1a 1.1b
	$\frac{1}{2}\pi\times 4^2-\left(\frac{1}{2}\times 4^2\times\frac{\pi}{3}-\frac{1}{2}\times 4^2\times\sin\frac{\pi}{3}\right)$ or $\frac{1}{2}\times 4^2\times\frac{2\pi}{3}+\frac{1}{2}\times 4\times 2\sqrt{3}$ or	dM1	3.1a

	$\frac{1}{2}\times 4^2\times\frac{2\pi}{3}+\frac{1}{2}\times 4\times 4\times\frac{\sqrt{3}}{2}$		
	$=\frac{16}{3}\pi+4\sqrt{3}$	A1	1.1b
		(4)	
	Alternative 3 for part (c) Polar coordinates		
	$(x-4)^2+y^2=16,\ \Rightarrow r^2=8r\cos\theta\Rightarrow r=8\cos\theta$ Area = $\int\frac{1}{2}r^2\ \mathrm{d}\theta=\int\frac{1}{2}.64\cos^2\theta\ \mathrm{d}\theta=\int(16+16\cos 2\theta)\mathrm{d}\theta$	M1 A1	3.1a 1.1b
	$=\left[16\theta+8\sin 2\theta\right]_0^{\frac{\pi}{3}}=...$	dM1	3.1a
	$=\frac{16}{3}\pi+4\sqrt{3}$	A1	1.1b
		(4)	
			(9 marks)

Notes

(c)

This is not for finding their shaded area of their diagram in part (a) but is for a correct process for finding the correct required area.

M1: Correct strategy for identifying both coordinates of the point of intersection.
Award for substituting a line of the form $y=kx$ into the equation of a circle $(x-4)^2+y^2=16$ then proceeds to find a value for x from a quadratic, where $x\neq 0$ and then find their y coordinate.
A1: Correct coordinates, either written separately, or as $(2,2\sqrt{3})$ or $2+2\sqrt{3}$ i
dM1: Fully correct strategy for the area, must be consistent with a half line making an angle of $\frac{\pi}{3}$ with the real axis.
Can be found by subtracting the area of a segment from the area of a semicircle or by adding the area of a sector to the area of a triangle.
The candidate may do combinations of semicircles, triangles and sectors so look carefully for the method.
A1: Correct answer in the required form.

Alternative 1

M1: Deduces that since the half line makes an angle of $\frac{\pi}{3}$ with the real axis, the horizontal and vertical distances from the origin to the point of intersection are $4\cos\frac{\pi}{3}$ and $4\sin\frac{\pi}{3}$.

A1: $4\cos\frac{\pi}{3}$ and $4\sin\frac{\pi}{3}$ oe are seen or used in their workings
dM1: Fully correct strategy for the area, must be consistent with a half line making an angle of $\frac{\pi}{3}$ with the real axis.
Can be found by subtracting the area of a segment from the area of a semicircle or by adding the area of a sector to the area of a triangle.

The candidate may do combinations of semicircles, triangles and sectors so look carefully for the method.
A1: Correct answer in the required form.

Alternative 2
M1: Alternatively deduces that the angle implies that there is an equilateral triangle of radius 4
A1: 4 and $\sin\frac{\pi}{3}$ are seen or used in their workings
dM1: Fully correct strategy for the area, must be consistent with a half line making an angle of $\frac{\pi}{3}$ with the real axis.
Can be found by subtracting the area of a segment from the area of a semicircle or by adding the area of a sector to the area of a triangle.
The candidate may do combinations of semicircles, triangles and sectors so look carefully for the method.
A1: Correct answer in the required form.

Alternative 3: using polar coordinates
M1: Achieves a polar equation of the form $r = k\cos\theta$ and uses $\left(\frac{1}{2}\right)\int r^2\,\mathrm{d}\theta$ to obtain $k\int\cos^2\theta\,\mathrm{d}\theta$
A1: Obtains $\int(16+16\cos 2\theta)\,\mathrm{d}\theta$ oe, $\mathrm{d}\theta$ may be missing
dM1: Integrates to obtain an expression of the form $a\theta + b\sin 2\theta$, substitutes in limits of 0 and $\frac{\pi}{3}$ and subtracts. If they reach a correct answer with no integration seen then withhold this mark.
A1: Correct answer in the form required

e.g. $(x-4)^2 + y^2 = 16, \;\Rightarrow r^2 = 8r\cos\theta \Rightarrow r = 8\cos\theta$

Area = $\int\frac{1}{2}r^2\mathrm{d}\theta = \int\frac{1}{2}.64\cos^2\theta\;\;\mathrm{d}\theta = \int\frac{32(1+\cos 2\theta)}{2}\mathrm{d}\theta = \int(16+16\cos 2\theta)\,\mathrm{d}\theta = [16\theta + 8\sin 2\theta]$

$[16\theta + 8\sin 2\theta]_0^{\frac{\pi}{3}} = \left[16\left(\frac{\pi}{3}\right) + 8\sin 2\left(\frac{\pi}{3}\right)\right] - [0] = \frac{16}{3}\pi + 4\sqrt{3}$

Question	Scheme	Marks	AOs
6(a)	$2m^2+5m+2=0 \Rightarrow m=-\frac{1}{2},-2$	M1	3.4
	$x=A\mathrm{e}^{-0.5t}+B\mathrm{e}^{-2t}$	A1	1.1b
	PI is of the form $x = pt + q$	B1	1.1b
	$\frac{\mathrm{d}x}{\mathrm{d}t}=p,\ \frac{\mathrm{d}^2x}{\mathrm{d}t^2}=0 \Rightarrow 5p+2pt+2q=4t+12 \Rightarrow p=...,q=...$	M1	3.4
	$p=2,q=1$	A1	1.1b
	$x=A\mathrm{e}^{-0.5t}+B\mathrm{e}^{-2t}+2t+1$	A1ft	1.1b
		(6)	
(b)	$t=0,\ x=3 \Rightarrow 3=A+B+1$	M1	3.4
	$\frac{\mathrm{d}x}{\mathrm{d}t}=-0.5A\mathrm{e}^{-0.5t}-2B\mathrm{e}^{-2t}+2$ $t=0,\ \frac{\mathrm{d}x}{\mathrm{d}t}=-2 \Rightarrow -\frac{1}{2}A-2B+2=-2$ $\Rightarrow A=...,\ B=...$	M1	3.4
	$x=2\mathrm{e}^{-2t}+2t+1$	A1	1.1b
		(3)	
(c)(i)	$\frac{\mathrm{d}x}{\mathrm{d}t}=-4\mathrm{e}^{-2t}+2$	M1	3.1b
	$\mathrm{e}^{-2t}=\frac{1}{2} \Rightarrow -2t=\ln\frac{1}{2} \Rightarrow t=\frac{1}{2}\ln 2$	dM1	2.1
	$x=2\mathrm{e}^{-2t}+2t+1=1+\ln 2+1=2+\ln 2$ *	A1*	1.1b
(c)(ii)	$\frac{\mathrm{d}^2x}{\mathrm{d}t^2}=8\mathrm{e}^{-2t}$ is > 0 for all values of t so distance is a minimum	B1ft	2.4
		(4)	
(d)	Examples: For large values of t, $\left[\mathrm{e}^{-2t}\to 0 \Rightarrow\right] x \to 2t+1$ so constant speed For large values of t, $\left[\mathrm{e}^{-2t}\to 0 \Rightarrow\right] \frac{\mathrm{d}x}{\mathrm{d}t} \to 2$ so constant speed For large values of t, $\left[\mathrm{e}^{-2t}\to 0 \Rightarrow\right] \frac{\mathrm{d}^2x}{\mathrm{d}t^2} \to 0$ so constant speed Conclusion: so the model is suitable	B1ft	3.2b
		(1)	
		(14 marks)	

Notes

(a)

M1: Attempts to solve $2m^2+5m+2=0$, usual rules apply for solving a quadratic equation.

A1: Correct CF. Do not need "$x =$" here, but must be in terms of t

B1: Correct form for the PI i.e. $x = pt + q$ or $x = at^2 + bt + c$

M1: Differentiates their PI (of the forms $x = pt + q$ or $x = at^2 + bt + c$) twice and substitutes into the given differential equation finding values for their constants to obtain a PI of the form $pt + q$, $p, q \neq 0$

A1: Correct PI

A1ft: Correct GS or correct ft GS, which is the sum of their CF and PI. This is dependent on achieving both previous M marks. **Must have $x =$** and their GS must be in terms of t.

(b)
M1: Substitutes $x = 3$ when $t = 0$ into their GS to establish an equation in A and B, allow minor slips if the intention is clear.

M1: Differentiates their answer to part (a) which must be in terms of t only, and sets $= -2$ with $t = 0$ to establish another equation in A and B and solves simultaneously to find A and B.
Do not be concerned about how their simultaneous equations are solved; award this mark if they then go on to write values for A and B.
Functions which require the use of product rule or trigonometric functions must be differentiated appropriately.

A1: Correct PS. **Need "x ="** here, their answer must be in terms of t and no other variable.

(c) Mark (i) and (ii) together but only award for work done in (c)
(i)
M1: Differentiates their Particular Solution of the form $x = ae^{-kt} + bt + c$ where c may be 0 to obtain an expression of the form $\mathrm{C}e^{-kt} + D$

dM1: Solves an equation of the form $\mathrm{C}e^{-kt} + D = 0, \quad C \times D < 0$, to obtain $t = -\frac{1}{k}\ln\left(\frac{-D}{C}\right)$

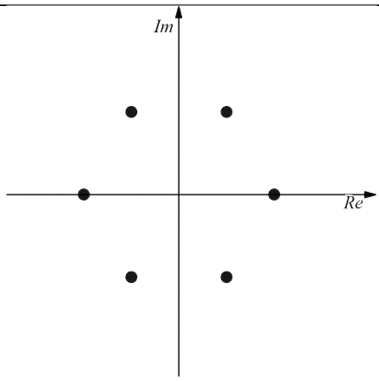
A1*: Substitutes $t = \frac{1}{2}\ln 2$ oe to obtain the printed answer with no errors.

Condone going from $2e^{-2\left(\frac{1}{2}\ln 2\right)} + 2\left(\frac{1}{2}\ln 2\right) + 1 = 2 + \ln 2$

(c)(ii)
B1ft: Obtains a second derivative of the form $\frac{d^2x}{dt^2} = \lambda e^{-\mu t}, \quad \lambda, \mu > 0$ **and** makes a conclusion e.g.

- $\frac{d^2x}{dt^2} > 0$ (for all values of t) hence minimum.
- or substitutes their value of t (even if incorrect) and states $\frac{d^2x}{dt^2} > 0$ hence minimum

(d)
B1ft: Dependent on having obtained a Particular Solution of the form $f(t) + bt + c$ where f(t) only has terms in ae^{-kt} where $k > 0$ and $a, b \neq 0$
This mark is awarded for the candidate demonstrating that in the model for "large values" or "as $t \to \infty$", the value of their $e^{-kt} \to 0$, so there is constant speed **and** states that the model is suitable, or equivalent statement. (See scheme for examples)

Question	Scheme	Marks	AOs
7(a)	$z = e^{\frac{k\pi}{3}i}, \quad k = 0, 1, 2, 3, 4, 5$	M1 A1	1.1b 1.1b
		(2)	
(b)		B1 dB1	2.2a 1.1b
		(2)	
(c)	e.g. $\left(\sqrt{3}+i\right)^6 = \left(2e^{\frac{\pi}{6}i}\right)^6 = 64e^{i\pi} = -64*$ **or** $\left[2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)\right]^6 = 2^6(\cos\pi + i\sin\pi) = 64(-1) = -64*$ **or** $\left(\sqrt{3}+i\right)^6 = \left(\sqrt{3}\right)^6 + 6\left(\sqrt{3}\right)^5 i - 15\left(\sqrt{3}\right)^4 - 20\left(\sqrt{3}\right)^3 i + 15\left(\sqrt{3}\right)^2 + 6\sqrt{3}i + i^6$ $= 27 - 135 + 45 - 1 = -64*$ **or** $\left(\sqrt{3}+i\right)^6 = = 27 + 54\sqrt{3}i + 135i^2 + 60\sqrt{3}i^3 + 45i^4 + 6\sqrt{3}i^5 + i^6$ $= 27 + 54\sqrt{3}i - 135 - 60\sqrt{3}i + 45 + 6\sqrt{3}i - 1 = -64*$	M1 A1*	1.1b 2.1
		(2)	
(d)	$r = 2$	B1	2.2a
	$z = 2e^{\frac{\pi}{6}i} \times e^{\frac{k\pi}{3}i}, \quad k = 0, 1, 2, 3, 4, 5$	M1	3.1a
	$z = 2e^{\left(\frac{\pi}{6}+\frac{k\pi}{3}\right)i}, \quad k = 0, 1, 2, 3, 4, 5$	A1	1.1b
		(3)	
			(9 marks)

Notes

(a)

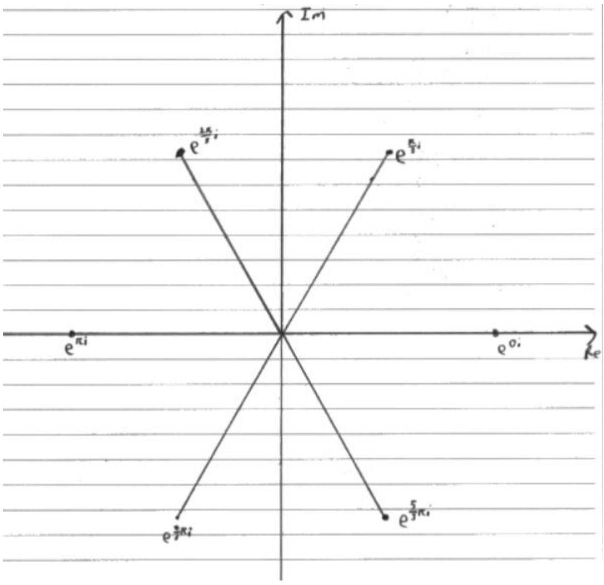
M1: For sight of $e^{\frac{k\pi}{3}i}$
Accept any value for *k*

A1: All six roots fully defined as shown or listed separately with their values of θ within the given range with no incorrect or extra values. Ensure i and π are present in each term.

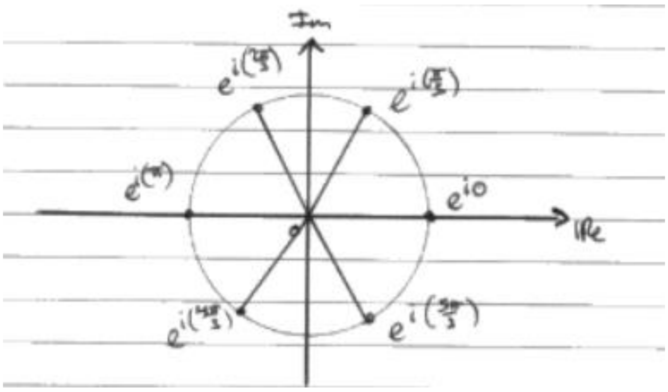
Note: Roots if listed are $e^0, e^{\frac{\pi}{3}i}, e^{\frac{2\pi}{3}i}, e^{\pi i}, e^{\frac{4\pi}{3}i}, e^{\frac{5\pi}{3}i}$, condone 1 for e^0 and/or -1 for $e^{\pi i}$

(b)
B1: Plots 6 points that form a hexagon, with a point on the positive real axis and a point on the negative real axis, and one point in each quadrant.
Do not be concerned about the position of each point from the centre, however the sketch must convey a hexagon.
dB1: The points form a hexagon, centre the origin (see diagram), axes need not be labelled.
Look for the axes acting as lines of symmetry.
(Drawing line/vectors to each point is acceptable but not necessary for either mark)

Examples



B1dB1



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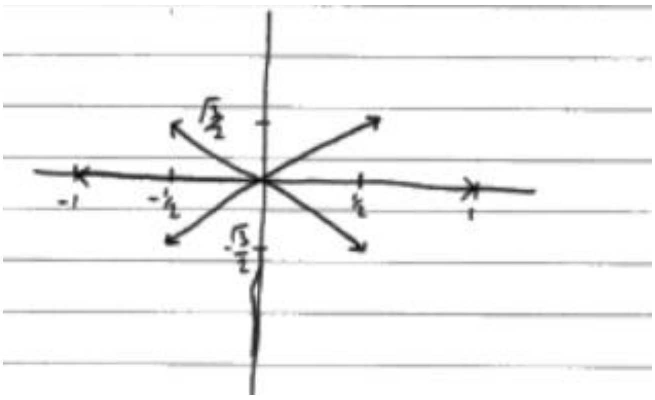

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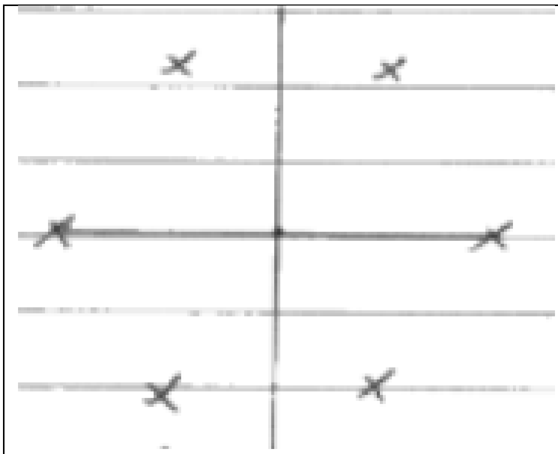

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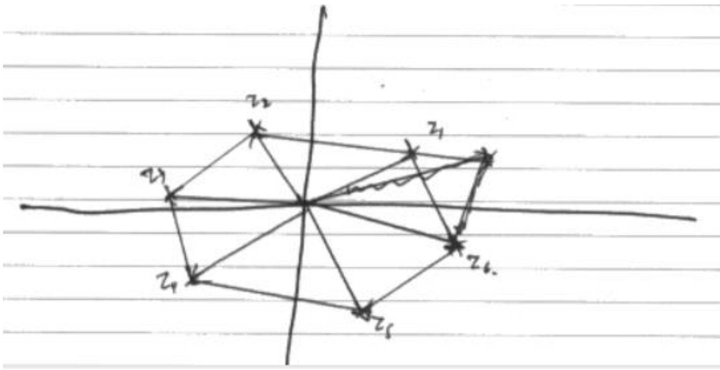
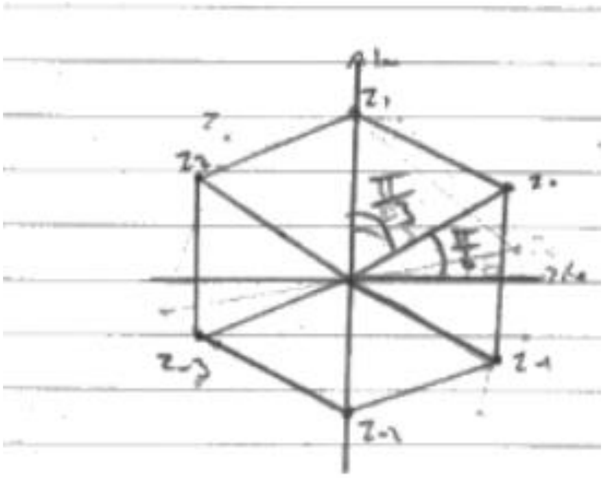


B0dB0



B0dB0

(c)

M1: Converts $\sqrt{3}+\mathrm{i}$ to polar form to obtain $r\mathrm{e}^{\mathrm{i}\theta}$ with at least $r = 2$ or $\theta = \dfrac{\pi}{6}$ and applies the power of 6 correctly to obtain $r^6\mathrm{e}^{6\theta\mathrm{i}}$

A1*: Obtains the given answer with sufficient working shown.

As a minimum need to see $2^6\mathrm{e}^{\frac{6\pi i}{6}} = -64$ or $2^6\mathrm{e}^{\pi\mathrm{i}} = -64$

If r = −2 is seen in their workings withhold this mark.

OR

M1: Converts $\sqrt{3}+\mathrm{i}$ to modulus-argument form $r(\cos\theta + \mathrm{i}\sin\theta)$ with at least $r = 2$ or $\theta = \dfrac{\pi}{6}$ and applies the power of 6 correctly to obtain $r^6(\cos 6\theta + \mathrm{i}\sin 6\theta)$

A1*: Obtains the given answer with sufficient working shown.

OR

M1: Attempts to expand $\left(\sqrt{3}+\mathrm{i}\right)^6$ fully using an attempt at the binomial expansion. Must have 7 terms for $(a+b)^n$ and correct binomial coefficients with $a = \sqrt{3}$, $b = \mathrm{i}$ and $n = 6$

A1*: Obtains the given answer with at least one intermediate line.

OR

M1: Attempts the full expansion of $\left(\sqrt{3}+\mathrm{i}\right)^6 = (\sqrt{3}+\mathrm{i})(\sqrt{3}+\mathrm{i})(\sqrt{3}+\mathrm{i})\ldots.(\sqrt{3}+\mathrm{i}) =$

There must be no brackets, no irrational numbers and no terms in i in their simplified answer.

A1*: Obtains the given answer with sufficient working shown including correct full expansion, with at least one intermediate line.

(d)

B1: Deduces $r = 2$ (only)

M1: Obtains at least one value of z in the form $r\mathrm{e}^{\mathrm{i}\theta}$ with their consistent value of r, and θ taking one of

$\left\{\frac{\pi}{6},\frac{\pi}{2},\frac{5\pi}{6},\frac{7\pi}{6},\frac{3\pi}{2},\frac{11\pi}{6}\right\}$

A1: For $2\mathrm{e}^{\frac{\pi}{6}\mathrm{i}},2\mathrm{e}^{\frac{\pi}{2}\mathrm{i}},2\mathrm{e}^{\frac{5\pi}{6}\mathrm{i}},2\mathrm{e}^{\frac{7\pi}{6}\mathrm{i}},2\mathrm{e}^{\frac{3\pi}{2}\mathrm{i}},2\mathrm{e}^{\frac{11\pi}{6}\mathrm{i}}$ with no incorrect or extra values. Accept unsimplified arguments such as having a solution of $2\mathrm{e}^{\frac{9\pi}{6}\mathrm{i}}$. Ensure i and π are present in each term.

Accept $2\mathrm{e}^{\frac{\pi}{2}\mathrm{i}}$ as $2\mathrm{i}$ and $2\mathrm{e}^{\frac{3\pi}{2}\mathrm{i}}$ as $-2\mathrm{i}$

Question	Scheme	Marks	AOs
8(a)	$\lvert\mathbf{A}\rvert = 3(6-3)-1(6-k)-1(3-k) = 2k$	B1	1.1b
	Cofactors $\begin{pmatrix} 3 & k-6 & 3-k \\ -9 & 18+k & k-9 \\ 2 & -4 & 2 \end{pmatrix}$ or Transpose of matrix of minors $\begin{pmatrix} 3 & 9 & 2 \\ 6-k & 18+k & 4 \\ 3-k & 9-k & 2 \end{pmatrix}$	M1	2.1
	$\mathbf{A}^{-1} = \frac{1}{2k}\begin{pmatrix} 3 & -9 & 2 \\ k-6 & 18+k & -4 \\ 3-k & k-9 & 2 \end{pmatrix}$	dM1 A1	1.1b 1.1b
		(4)	
(b)	$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{2k}\begin{pmatrix} 3 & -9 & 2 \\ k-6 & 18+k & -4 \\ 3-k & k-9 & 2 \end{pmatrix}\begin{pmatrix} 3 \\ 1 \\ 6 \end{pmatrix} = \ldots$	M1	1.1a
	Any 2 of $x = \frac{6}{k}$, $y = \frac{2k-12}{k}$, $z = \frac{6-k}{k}$	A1	2.1
	$\left(\frac{6}{k}, \frac{2k-12}{k}, \frac{6-k}{k}\right)$ **or** $x = \frac{6}{k}$, $y = \frac{2k-12}{k}$, $z = \frac{6-k}{k}$ **or** $x = \frac{6}{k}$, $y = 2 - \frac{12}{k}$, $z = \frac{6}{k} - 1$	A1	2.5
		(3)	
			(7 marks)

Notes

(a)

If **no attempt** at part (a) has been made you may award marks for work seen in part (b) for finding the inverse.

B1(M1 on epen): Correct determinant of $2k$

M1: Starts the process of finding the inverse and obtains at least 6 correct elements of cofactors
Alternatively transposes their matrix of minors and obtains at least 6 correct elements.

dM1: A complete recognisable method to find the inverse including dividing by the determinant
Allow minor slips if the process is clearly correct.

A1: Correct inverse.

(b)

M1: Attempts $\mathbf{A}^{-1}\begin{pmatrix}3\\1\\6\end{pmatrix}$ with their $\mathbf{A}^{-1}$ which must be in terms of k,

to obtain at least one of $x=$, $y=$ or $z=$ which may be seen embedded in a column vector, simplified or unsimplified, the determinant may be outside their column vector.

Condone a slip in copying $\begin{pmatrix}3\\1\\6\end{pmatrix}$ e.g. $\begin{pmatrix}1\\3\\6\end{pmatrix}$ or $\begin{pmatrix}3\\1\\2\end{pmatrix}$

A1(M1 on epen): Two correct expressions for x, y or z simplified or unsimplified, which may appear in a column vector, determinant **cannot** be outside the vector.

A1: Correct coordinates in simplest form. Allow e.g. $y=2-\frac{12}{k}$

Their final answer must be written as coordinates and not as a column vector but can be written as $x=$, $y=$ and $z=$

Alternative (**using algebraic method for simultaneous equations**):

M1: Solves simultaneously to obtain at least one of $x=$, $y=$ or $z=$ must be in terms of k.

A1(M1 on epen): Two correct expressions for x, y or z simplified or unsimplified.

A1: Correct coordinates in simplest form. Allow e.g. $y=2-\frac{12}{k}$

Their final answer must be written as coordinates and not as a column vector but can be written as $x=$, $y=$ and $z=$

e.g.

Eliminates z and achieves $4x+2y=4$ and $(6-k)x+3y=0$

Uses $12x+6y=12$ and $(12-2\text{k})x+6y=0$ to produce $2kx=12 \Rightarrow x=\frac{6}{k}$

Question	Scheme	Marks	AOs
9(a)	$a=2$ **or** $b=7$	B1	3.3
	$a=2$ **and** $b=7$	B1	3.3
		(2)	
(b)	$V=(\pi)\int x^2\frac{\mathrm{d}y}{\mathrm{d}t}\mathrm{d}t=\int(2+3\sin 2t)^2(-7\sin t)\mathrm{d}t$	M1	3.4
	$=-7(\pi)\int(4\sin t+12\sin 2t\sin t+9\sin^2 2t\sin t)\mathrm{d}t$ $=-7(\pi)\int(4\sin t+24\sin^2 t\cos t+36\sin^3 t\cos^2 t)\mathrm{d}t$	M1	3.1a
	$=-7(\pi)\left[-4\cos t+8\sin^3 t-12\cos^3 t+\frac{36}{5}\cos^5 t\right]$	A1ft A1	1.1b 1.1b
	Cylinder volume is $\pi\times 2^2\times 4.5=(18\pi)$	B1	3.4
	Total Volume = V + cylinder volume $=-7\pi\left[-4\cos t+8\sin^3 t-12\cos^3 t+\frac{36}{5}\cos^5 t\right]_{\frac{\pi}{2}}^{0}+\pi\times 2^2\times 4.5$ or $=7\pi\left[-4\cos t+8\sin^3 t-12\cos^3 t+\frac{36}{5}\cos^5 t\right]_{0}^{\frac{\pi}{2}}+\pi\times 2^2\times 4.5$ (NB this is $\frac{588}{5}\pi+18\pi$)	ddM1	3.4
	$426\ (\mathrm{cm}^3)$	A1	2.2b
		(7)	
(c)	Any one of e.g. The vase may not be completely smooth The vase may not be symmetrical The measurements may not be accurate The equation of the curve may not be a suitable model The thickness of the sides has not been considered Accept the base may have a dimple in it (the base may not be completely flat)	B1	3.5b
		(1)	
		(10 marks)	

Notes

(a)
B1: Uses the model to obtain a correct value for *a* or *b*
B1: Uses the model to obtain correct values for *a* and *b*

(b)

M1: Uses the parametric curve for the model and applies (π) $\int x^2 \frac{dy}{dt} dt$

The π symbol may be missing here. Also do not be concerned about a missing dt at the end of their integral.

Must see an attempt at squaring x and finding $\frac{dy}{dt}$ where $x = (a + 3\sin 2t)$ and $\frac{dy}{dt} = k\sin t$ with their a and b or the letters a and b in their integral.

M1: Expands and makes progress to an integrable form by applying sin2t = 2sin$t$ cos$t$ at least once

Integrals in terms of a and b will be equivalent to:

$$\pm(\pi)\int (a^2 b \sin t + 6ab \sin 2t \sin t + 9b \sin^2 2t \sin t)\, dt$$

$$= \pm(\pi)\int a^2 b \sin t + 12ab \sin^2 t \cos t + 36b \sin^3 t \cos^2 t\, dt$$

Integrals with correct values of a and b will be equivalent to:

$$\pm(\pi)\int \left(28\sin t + 84\sin 2t \sin t + 63\sin^2 2t \sin t\right) dt$$

$$= \pm(\pi)\int \left(28\sin t + 168\sin^2 t\cos t + 252\sin^3 t\cos^2 t\right) dt$$

A1ft: (dependent on **both** M marks) At least 2 terms integrated correctly; follow through their a and b, however their a and b must now be numerical.

$$\pm(\pi)\left[-a^2 b\cos t + 4ab\sin^3 t - \frac{36}{3}b\cos^3 t + \frac{36b}{5}\cos^5 t\right]$$

$$= \pm(\pi)\left[28\cos t - 56\sin^3 t + 84\cos^3 t - \frac{252}{5}\cos^5 t\right]$$

A1: All correct from correct values for a and b

Their integral will be equivalent to:

$$= \pm(\pi)\left[28\cos t - 56\sin^3 t + 84\cos^3 t - \frac{252}{5}\cos^5 t\right]$$

B1: Uses the model to deduce the correct volume of the cylinder; need not be simplified.

ddM1: Dependent on **both** previous M marks. Fully correct strategy for the volume. Applies the correct limits (either way round) and adds this to the volume of the cylinder which must have been obtained from $\pi \times 2^2 \times 4.5$ but condone $\pi \times 4^2 \times 4.5$

Both volumes must be positive when combined.

A1: Correct volume, allow awrt 426 (cm^3), units are not required but if any are given, they should be correct.

(c)

B1: See scheme. Award for a correct statement. If there is more than one statement ignore any incorrect statements as long as there are no statements that contradict their correct statement.

Comments relating to facts about the shape that are **not** worthy of marks include:

- the cross section is not a circle
- The vase is not solid
- the cylinder might not be vertical
- it depends on the material used to make the vase
- the vase may not have the same density throughout

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